CAMBRIDGE

Sixth Term Examination Paper [STEP]

Mathematics 2 [9470]

2024

Examiners' Report

Mark Scheme

STEP MATHEMATICS 2

2024

Mark Scheme

Question			Answer	Mark
1	**i**		The two sums are $\frac{1}{2}(n+k)(2c+(n+k-1))$ and $\frac{1}{2}n(2(c+n+k)+(n-1))$	**B1**
			Difference simplifies to $\frac{1}{2}(2ck+k^2-2n^2-k)$	**M1**
			Two sums are equal if and only if the difference is 0 if and only $2n^2+k=2ck+k^2$	**A1**
				[3]
	ii	**a**	If $k=1$, require $n^2=c$. Any value of n is possible.	**B1**
		b	If $k=2$, require $n^2=2c+1$	**M1**
			n can be any odd value,	**A1**
			and $c=\frac{n^2-1}{2}$	**A1**
				[4]
	iii		If $k=4$, require $n^2=4c+6$	**B1**
			RHS has a factor of 2, but not a factor of 4 …	**E1**
			… so cannot be a square.	**E1**
				[3]
	iv	**a**	If $c=1$, require $2n^2=k(k+1)$:	
			$k=1, n=1$	**B1**
			$k=8, n=6$	**B1**
				[2]

Question	Answer	Mark
1 iv b		
	Since (N, K) is a solution: $2N^2 + K = 2K + K^2$ or $2N^2 = K(K+1)$	B1
	$2(3N + 2K + 1)^2 + (4N + 3K + 1) =$ $18N^2 + 8K^2 + 3 + 24NK + 16N + 11K$ OR $2(3N + 2K + 1)^2 =$ $18N^2 + 8K^2 + 2 + 24NK + 12N + 8K$	M1
	$2(4N + 3K + 1) + (4N + 3K + 1)^2 =$ $16N^2 + 9K^2 + 3 + 24NK + 16N + 12K$ OR $(4N + 3K + 1)(4N + 3K + 2) =$ $16N^2 + 9K^2 + 2 + 24NK + 12N + 9K$	M1
	Difference between the two expressions that use = $2N^2 - K^2 - K$	M1
	$= 0$, so $N' = (3N + 2K + 1)$ is a possible value for n, with $K' = (4N + 3K + 1)$ as the corresponding value of k.	A1
		[5]
c	Use of recurrence with one of the pairs found in part (iv)(a)	M1
	$k = 49, n = 35$	A1
	$k = 288, n = 204$	A1
		[3]

Question	Answer	Mark
2 i	$(8+x^3)^{-1} = \frac{1}{8}\left(1+\frac{x^3}{8}\right)^{-1}$	
	$= \frac{1}{8}\left(1 - \frac{x^3}{8} + \frac{x^6}{64} - \frac{x^9}{512} + \cdots\right)$	**M1**
	$= \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{3k}$	**A1**
	$\int_0^1 \frac{x^m}{8+x^3}\,dx = \int_0^1 \frac{1}{8}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3k}}x^{m+3k}\,dx$	**M1**
	$= \frac{1}{8}\sum_{k=0}^{\infty}\left[\frac{(-1)^k}{2^{3k}}\frac{x^{m+3k+1}}{m+3k+1}\right]_0^1$	**A1**
	$= \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\frac{1}{m+3k+1}\right)$	**A1**
		[5]
ii	$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{1}{3k+3}\right) = \int_0^1 \frac{x^2}{8+x^3}\,dx$ $\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{-2}{3k+2}\right) = \int_0^1 \frac{-2x}{8+x^3}\,dx$ $\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{4}{3k+1}\right) = \int_0^1 \frac{4}{8+x^3}\,dx$	**M1**
	$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{1}{3k+3} - \frac{2}{3k+2} + \frac{4}{3k+1}\right)$ $= \int_0^1 \frac{x^2-2x+4}{8+x^3}\,dx$	**A1**
	$= \int_0^1 \frac{x^2-2x+4}{(x+2)(x^2-2x+4)}\,dx$	**M1**
	$= \int_0^1 \frac{1}{x+2}\,dx$	**A1**
	$= [\ln(x+2)]_0^1 = \ln\left(\frac{3}{2}\right)$	**B1**
		[5]
iii	$\frac{72(2k+1)}{(3k+1)(3k+2)} = \frac{24}{3k+1} + \frac{24}{3k+2}$	**M1** **A1**
	$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\frac{72(2k+1)}{(3k+1)(3k+2)} = \int_0^1 \frac{24x+24}{8+x^3}\,dx$	**A1**
	$= \int_0^1 \frac{2(x+8)}{x^2-2x+4} - \frac{2}{x+2}\,dx$	**M1** **A1**
	$= \int_0^1 \frac{2(x-1)}{x^2-2x+4} + \frac{18}{x^2-2x+4} - \frac{2}{x+2}\,dx$	**M1**
	$= [\ln(x^2-2x+4)]_0^1 \ldots$	**M1**
	$\ldots + \left[6\sqrt{3}\arctan\left(\frac{x-1}{\sqrt{3}}\right)\right]_0^1 \ldots$	**M1**
	$\ldots - [2\ln(x+2)]_0^1$	**A1**
	$= \ln 3 - \ln 4 - 2\ln 3 + 2\ln 2 + 6\sqrt{3}\cdot\frac{\pi}{6}$ $= \pi\sqrt{3} - \ln 3$	**A1**
		[10]

Question	Answer	Mark
3 i	Gradient of NP is $\frac{\sin\theta}{1+\cos\theta}$ $\left(=\frac{y}{1}\right)$	M1
	$y=\dfrac{2\sin\left(\frac{1}{2}\theta\right)\cos\left(\frac{1}{2}\theta\right)}{1+2\cos^2\left(\frac{1}{2}\theta\right)-1}$	M1
	$=\tan\left(\frac{1}{2}\theta\right)$	A1
		[3]
ii a	$f_1(q)=\dfrac{\tan\frac{1}{4}\pi+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{4}\pi\tan\frac{1}{2}\theta}$	B1 M1
	$=\tan\frac{1}{2}\left(\theta+\frac{1}{2}\pi\right)$	A1
		[3]
ii b	If the coordinates of P_1 are $(\cos\psi,\sin\psi)$: $f(q_1)=\tan\left(\frac{1}{2}\psi\right)=\tan\frac{1}{2}\left(\theta+\frac{1}{2}\pi\right)$	M1
	P_1 is the image of P under rotation anticlockwise through a right angle about O.	B1 B1
		[3]
iii a	$f_2(q)=\tan\frac{1}{2}\left(\theta+\frac{1}{3}\pi\right)$	M1
	$=\dfrac{\tan\left(\frac{1}{6}\pi\right)+q}{1-q\tan\left(\frac{1}{6}\pi\right)}$	A1
	$=\dfrac{1+\sqrt{3}q}{\sqrt{3}-q}$	A1
		[3]
iii b	$f_3(q)=f_1(-q)$, so	M1
	$f_3(q)=\tan\frac{1}{2}\left(\frac{1}{2}\pi-\theta\right)$	A1
	So the coordinates of P_3 are $(\sin\theta,\cos\theta)$	M1
	P_3 is the image of P under reflection in $y=x$	A1
		[4]
iii c	P_4 is the image of P under the following sequence of transformations: Rotation anticlockwise through $\frac{1}{3}\pi$ Reflection in $y=x$ Rotation clockwise through $-\frac{1}{3}\pi$	M1 A1
	A point is invariant under this transformation if its image under the rotation anticlockwise through $\frac{1}{3}\pi$ lies on the line $y=x$	M1
	… making an angle of $-\frac{\pi}{12}$ with the positive x-axis	A1
		[4]

Question			Answer	Mark
4	i	a	$\boldsymbol{b}$ is a linear combination of $\boldsymbol{x}$ and $\boldsymbol{y}$, so it must lie in the plane OXY	B1
			$\boldsymbol{b}\cdot\boldsymbol{x} = (\lvert\boldsymbol{x}\rvert\boldsymbol{y} + \lvert\boldsymbol{y}\rvert\boldsymbol{x})\cdot\boldsymbol{x} = \lvert\boldsymbol{x}\rvert\boldsymbol{y}\cdot\boldsymbol{x} + \lvert\boldsymbol{y}\rvert\lvert\boldsymbol{x}\rvert^2$	M1
			If θ is the angle between $\boldsymbol{x}$ and $\boldsymbol{b}$, then $\cos\theta = \frac{\boldsymbol{b}\cdot\boldsymbol{x}}{\lvert\boldsymbol{b}\rvert\lvert\boldsymbol{x}\rvert} = \frac{\boldsymbol{x}\cdot\boldsymbol{y} + \lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert}{\lvert\boldsymbol{b}\rvert}$	M1
			Similarly, $\frac{\boldsymbol{b}\cdot\boldsymbol{y}}{\lvert\boldsymbol{b}\rvert\lvert\boldsymbol{y}\rvert} = \frac{\boldsymbol{x}\cdot\boldsymbol{y} + \lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert}{\lvert\boldsymbol{b}\rvert}$ so the angle between $\boldsymbol{b}$ and $\boldsymbol{y}$ is also θ.	A1
			Since $\boldsymbol{x}\cdot\boldsymbol{y} + \lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert > 0$, $\cos\theta > 0$ and so the angle is less than $90°$	E1
			A sketch to indicate why any other bisecting vector is a positive multiple of this.	E1
				[6]
	i	b	The vector $\overrightarrow{XB} = \lambda\boldsymbol{b} - \boldsymbol{x}$ must be parallel to the vector $\overrightarrow{XY} = \boldsymbol{y} - \boldsymbol{x}$.	
			For some μ: $\lambda\boldsymbol{b} - \boldsymbol{x} = \mu(\boldsymbol{y} - \boldsymbol{x})$	M1
			$\lambda(\lvert\boldsymbol{x}\rvert\boldsymbol{y} + \lvert\boldsymbol{y}\rvert\boldsymbol{x}) - \boldsymbol{x} = \mu(\boldsymbol{y} - \boldsymbol{x})$ $(\lambda\lvert\boldsymbol{y}\rvert + \mu - 1)\boldsymbol{x} = (\mu - \lambda\lvert\boldsymbol{x}\rvert)\boldsymbol{y}$	A1
			Since $\boldsymbol{x}$ and $\boldsymbol{y}$ are not parallel: $\lambda\lvert\boldsymbol{y}\rvert + \mu - 1 = 0$ $\mu - \lambda\lvert\boldsymbol{x}\rvert = 0$	E1
			$\lambda = \frac{1}{\lvert\boldsymbol{x}\rvert + \lvert\boldsymbol{y}\rvert}$	M1
			$\mu = \frac{\lvert\boldsymbol{x}\rvert}{\lvert\boldsymbol{x}\rvert + \lvert\boldsymbol{y}\rvert}$	M1
			So B divides XY in the ratio $\lvert\boldsymbol{x}\rvert : \lvert\boldsymbol{y}\rvert$	A1
				[6]
	i	c	If OB is perpendicular to XY: $\boldsymbol{b}\cdot(\boldsymbol{y} - \boldsymbol{x}) = 0$ $\lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert^2 + \lvert\boldsymbol{y}\rvert\boldsymbol{x}\cdot\boldsymbol{y} - \lvert\boldsymbol{x}\rvert\boldsymbol{y}\cdot\boldsymbol{x} - \lvert\boldsymbol{y}\rvert\lvert\boldsymbol{x}\rvert^2 = 0$ $(\lvert\boldsymbol{y}\rvert - \lvert\boldsymbol{x}\rvert)(\lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert + \boldsymbol{x}\cdot\boldsymbol{y}) = 0$	M1 A1
			$\lvert\boldsymbol{x}\rvert\lvert\boldsymbol{y}\rvert + \boldsymbol{x}\cdot\boldsymbol{y} > 0$ So $\lvert\boldsymbol{x}\rvert = \lvert\boldsymbol{y}\rvert$	A1
				[3]
	ii		Let $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ be the position vectors of P, Q and R respectively.	
			The bisecting vector of POQ is $\lvert\boldsymbol{p}\rvert\boldsymbol{q} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}$ The bisecting vector of QOR is $\lvert\boldsymbol{q}\rvert\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{q}$	
			If θ is the angle between these two vectors, then: $\cos\theta = \frac{(\lvert\boldsymbol{p}\rvert\boldsymbol{q} + \lvert\boldsymbol{q}\rvert\boldsymbol{p})\cdot(\lvert\boldsymbol{q}\rvert\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{q})}{\lvert\lvert\boldsymbol{p}\rvert\boldsymbol{q} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}\rvert\,\lvert\lvert\boldsymbol{q}\rvert\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{q}\rvert}$	M1
			$= \frac{\lvert\boldsymbol{p}\rvert\lvert\boldsymbol{q}\rvert\boldsymbol{q}\cdot\boldsymbol{r} + \lvert\boldsymbol{p}\rvert\lvert\boldsymbol{r}\rvert\lvert\boldsymbol{q}\rvert^2 + \lvert\boldsymbol{q}\rvert^2\boldsymbol{p}\cdot\boldsymbol{r} + \lvert\boldsymbol{q}\rvert\lvert\boldsymbol{r}\rvert\boldsymbol{p}\cdot\boldsymbol{q}}{\lvert\lvert\boldsymbol{p}\rvert\boldsymbol{q} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}\rvert\,\lvert\lvert\boldsymbol{q}\rvert\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{q}\rvert}$	M1
			$\cos\theta = \frac{\lvert\boldsymbol{q}\rvert(\lvert\boldsymbol{p}\rvert\lvert\boldsymbol{q}\rvert\lvert\boldsymbol{r}\rvert + \lvert\boldsymbol{p}\rvert\boldsymbol{q}\cdot\boldsymbol{r} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}\cdot\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{p}\cdot\boldsymbol{q})}{\lvert\lvert\boldsymbol{p}\rvert\boldsymbol{q} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}\rvert\,\lvert\lvert\boldsymbol{q}\rvert\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{q}\rvert}$	A1
			$\lvert\boldsymbol{p}\rvert\lvert\boldsymbol{q}\rvert\lvert\boldsymbol{r}\rvert + \lvert\boldsymbol{p}\rvert\boldsymbol{q}\cdot\boldsymbol{r} + \lvert\boldsymbol{q}\rvert\boldsymbol{p}\cdot\boldsymbol{r} + \lvert\boldsymbol{r}\rvert\boldsymbol{p}\cdot\boldsymbol{q}$ is symmetrical in $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$.	E1
			All other factors are strictly positive, so the sign will be the same for all three angles (and so the angles are either all acute, all right angles or all obtuse).	E1
				[5]

Question	Answer	Mark
5 i	$f_1(t) = (t+3)^2 + 2 = F_1(t+3)$	M1
	Since $t \mapsto t+3$ is a one-to-one correspondence on $\mathbb{Z}$, the functions have the same range.	A1
		[2]
ii	If there is a value that lies in both the range of f_1 and g_1, then there are integers s and t such that: $f_1(s) = (s+3)^2 + 2 = g_1(t) = (t-1)^2 + 4$	M1
	$(s+3)^2 - (t-1)^2 = 2$	A1
	which is not possible for any integers s and t.	E1
		[3]
iii	For any value that lies in both the range of f_2 and g_2, there are integers s and t such that: $f_2(s) = (s-1)^2 - 7 = g_2(t) = (t-2)^2 - 2$	M1
	$(s-1)^2 - (t-2)^2 = 5$ $(s+t-3)(s-t+1) = 5$	A1
	$s+t-3 = 1$ $s-t+1 = 5$ has solution $s = 4, t = 0$	M1
	$s+t-3 = -1$ $s-t+1 = -5$ has solution $s = -2, t = 4$	
	$s+t-3 = 5$ $s-t+1 = 1$ has solution $s = 4, t = 4$	
	$s+t-3 = -5$ $s-t+1 = -1$ has solution $s = -2, t = 0$	A1
	All cases lead to $f_2(s) = g_2(t) = 2$, so only 2 lies in the intersection between the ranges.	A1
		[5]
iv	$4(p^2+pq+q^2) = (p-q)^2 + 3(p+q)^2$ $p^2+pq+q^2 = (p+q)^2 - pq = (p-q)^2 + 3pq$ sufficient for M1	M1
	Therefore $p^2+pq+q^2 \geq 0$ for all real p and q.	A1
		[2]
	$f_3(s) = s^3 - 3s^2 + 7s = g_3(t) = t^3 + 4t - 6$	
	$(s-1)^3 = s^3 - 3s^2 + 3s - 1$, so $(s-1)^3 - t^3 + 4s - 4t = -7$	M1
	$(s-1-t)((s-1)^2 + (s-1)t + t^2) + 4(s-1-t) = -11$	M1
	$(s-1-t)((s-1)^2 + (s-1)t + t^2 + 4) = -11$	A1
	By the result at the start of part (iv): $((s-1)^2 + (s-1)t + t^2 + 4) \geq 4$ so the product can only be -1×11	M1 A1
	We have $s = t$ and so $s^2 - 2s + 1 + s^2 - s + s^2 + 4 = 11$ $3s^2 - 3s - 6 = 0$, so $3(s-2)(s+1) = 0$, giving $s = t = 2$ or $s = t = -1$	M1 A1
	So the intersection is $\{f_3(-1), f_3(2)\} = \{-11, 10\}$	A1
		[8]

Question	Answer	Mark
6 i	For $n = 0$: $$T_0 = \frac{1}{2^0}\binom{0}{0} = 1$$	B1
	Assume that the result is true for $n = k$: $T_k = \frac{1}{2^{2k}}\binom{2k}{k}$	
	$T_{k+1} = \frac{2(k+1)-1}{2(k+1)} \cdot \frac{1}{2^{2k}}\binom{2k}{k}$	M1
	$T_{k+1} = \frac{1}{2^{2k}} \cdot \frac{2k+1}{2(k+1)} \cdot \frac{(2k)!}{(k!)^2}$ $T_{k+1} = \frac{1}{2^{2k}} \cdot \frac{2k+1}{2(k+1)} \cdot \frac{(2k)!}{(k!)^2} \cdot \frac{2k+2}{2(k+1)}$ $T_{k+1} = \frac{1}{2^{2(k+1)}} \cdot \frac{(2(k+1))!}{((k+1)!)^2}$ $T_k = \frac{1}{2^{2(k+1)}}\binom{2(k+1)}{k+1}$	M1 A1
	Hence, by induction: $T_n = \frac{1}{2^{2n}}\binom{2n}{n}$	A1
		[5]
ii	$a_r = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2r-1}{2}\right)\frac{(-1)^r}{r!}$	M1 A1
	$a_{r-1} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2(r-1)-1}{2}\right)\frac{(-1)^{r-1}}{(r-1)!}$	M1
	Therefore, $a_r = a_{r-1} \cdot \left(-\frac{2r-1}{2}\right) \cdot \frac{-1}{r}$ $a_r = \frac{2r-1}{2r} a_{r-1}$	E1
	Since $a_0 = 1 = T_0$,	B1
	$a_r = T_r$ for $r = 0, 1, 2, \cdots$	A1
		[6]
iii	$$b_r = \frac{\left(\frac{3}{2} \cdot \frac{5}{2} \cdot \ldots \cdot \frac{(2r-1)}{2} \cdot \frac{(2r+1)}{2}\right)}{r!}$$	M1
	So, $\frac{b_r}{a_r} = 2r + 1$ Correctly argued for general terms	A1 E1
	$b_r = \frac{2r+1}{2^{2r}}\binom{2r}{r}$	A1
		[4]

Question	Answer	Mark
6 iv	$(1-x)^{-1} = \Sigma_{r=0}^{\infty} x^r$	**B1**
	$(1 + x + x^2 + \cdots)(a_0 + a_1 x + a_2 x^2 + \cdots) = (b_0 + b_1 x + b_2 x^2 + \cdots)$	**B1**
	The term in x^n on the LHS is: $1 \cdot a_n x^n + x \cdot a_{n-1} x^{n-1} + \cdots + x^n \cdot a_0$	**M1** **A1**
	Therefore, $b_n = \Sigma_{r=0}^{\mathrm{n}} a_r$ as required.	**A1**
		[5]

Question	Answer	Mark
7 i	Circles with centres at (1,0) and (-1,0),	B1
	both with radius 1	B1
		[2]
ii a	$y = k$ meets the curve when $(x^2 + k^2 - 2x)(x^2 + k^2 + 2x) = \frac{1}{16}$	
	$(x^2 + k^2 - 2x)(x^2 + k^2 + 2x) = (x^2 + k^2)^2 - 4x^2$	M1
	So $(x^2 + k^2)^2 - 4x^2 - \frac{1}{16} = 0$ $x^4 + 2x^2(k^2 - 2) + k^4 - \frac{1}{16} = 0$	A1
	$(x^2 + k^2 - 2)^2 + 4k^2 - \frac{65}{16} = 0$ $x^2 = 2 - k^2 \pm \sqrt{\frac{65}{16} - 4k^2}$	M1 A1
	If $k^2 > \frac{65}{64}$ there will be no roots If $k^2 = \frac{65}{64}$ there will be two roots	B1
	If $k^2 < \frac{65}{64}$, the smaller of the two values of x^2 is $2 - k^2 - \sqrt{\frac{65}{16} - 4k^2}$	M1
	$2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} = 0$ when $(2 - k^2)^2 = \frac{65}{16} - 4k^2$	
	$k^4 = \frac{1}{16}$ So there will be three roots if $k^2 = \frac{1}{4}$	A1
	There will be two roots if $0 \le k^2 < \frac{1}{4}$ There will be four roots if $\frac{1}{4} < k^2 < \frac{65}{64}$	A1
		[8]
ii b	Greatest possible *y*-coordinate is when $k^2 = \frac{65}{64}$	M1
	So $x^2 = 2 - k^2 = \frac{63}{64} < 1$ So these points are closer to the *y*-axis than those on C_1	A1
		[2]
ii c	If both expressions are negative, then the distance from (x, y) to the points $(1,0)$ and $(-1,0)$ would both be less than 1.	E1
	But the shortest distance between $(1,0)$ and $(-1,0)$ is 2, so this is not possible. Therefore, it is not possible for both expressions to be negative.	E1
	Since the product of the two expression is positive and they are not both negative, they must both be positive. Therefore, the distance between the point (x, y) and each of the points $(1,0)$ and $(-1,0)$ must be greater than 1, so the curve C_2 lies entirely outside the circle C_1	E1
		[3]

Question	Answer	Mark
7 ii d	Continuous curve outside the two circles of C_1	**G1**
	Symmetrical under reflection in *x* and *y* axes.	**G1**
	Intersections with *x*-axis at $x = \pm\frac{1}{2}\sqrt{8+\sqrt{65}}$	**G1**
	Intersections with *y*-axis at $y = \pm\frac{1}{2}$	**G1**
	Maxima and minima at $\left(\pm\frac{1}{8}\sqrt{63}, \pm\frac{1}{8}\sqrt{65}\right)$	**G1**
		[5]

Question	Answer	Mark
8 i	$\left(\sqrt{x_n}-\sqrt{y_n}\right)^2 = 2a(x_n,y_n)-2g(x_n,y_n)$ $= 2(x_{n+1}-y_{n+1})$	M1
	So $x_{n+1}-y_{n+1} \geq 0$ for $n \geq 0$	A1
	$y_0 < x_0$ is given Suppose that $y_k < x_k$: $\left(\sqrt{x_k}-\sqrt{y_k}\right)^2 > 0$ and so $x_{k+1}-y_{k+1} > 0$	
	Hence, by induction, $y_n < x_n$ for $n \geq 0$	A1
		[3]
	$y_{n+1} = \sqrt{x_n}\sqrt{y_n} > \sqrt{y_n}\sqrt{y_n} = y_n$	B1
	$x_{n+1} = \frac{1}{2}(x_n+y_n) < \frac{1}{2}(x_n+x_n) = x_n$	B1
	$y_n < x_n < x_0$ for $n \geq 0$, so the sequence is bounded above.	B1
	As shown above the sequence is increasing, so the result given at the start of the question applies. There is a value M such that $y_n \to M$ as $n \to \infty$	B1
		[4]
	$x_{n+1}-y_{n+1} = \frac{1}{2}\left(\sqrt{x_n}-\sqrt{y_n}\right)^2$ $< \frac{1}{2}\left(\sqrt{x_n}-\sqrt{y_n}\right)\left(\sqrt{x_n}+\sqrt{y_n}\right)$	M1
	$= \frac{1}{2}(x_n-y_n)$	
	$x_{n+1}-y_{n+1} = \frac{1}{2}\left(\sqrt{x_n}-\sqrt{y_n}\right)^2 > 0$, since $x_n \neq y_n$ for any value of n. Therefore $0 < x_{n+1}-y_{n+1} < \frac{1}{2}(x_n-y_n)$	A1
	Hence $x_n - y_n \to 0$ as $n \to \infty$	E1
	So $x_n \to M$ as $n \to \infty$	E1
		[4]

Question	Answer	Mark
8 ii	$\frac{dt}{dx} = \frac{1}{2}\left(1 + \frac{pq}{x^2}\right)$	M1
	Limits: As $x \to 0$, $t \to -\infty$ As $x \to \infty$, $t \to \infty$	E1
	$\frac{1}{4}(p+q)^2 + \frac{1}{4}\left(x - \frac{pq}{x}\right)^2 = \frac{1}{4x^2}(x^4 + (p^2+q^2)x^2 + p^2q^2)$	M1
	$= \frac{1}{4x^2}(x^2+p^2)(x^2+q^2)$	
	$pq + \frac{1}{4}\left(x - \frac{pq}{x}\right)^2 = \frac{1}{4x^2}(x^4 + 2pqx^2 + p^2q^2)$ $= \frac{1}{4x^2}(x^2+pq)^2$	A1
	So the integral becomes: $2\int_0^\infty \frac{1}{\sqrt{(x^2+p^2)(x^2+q^2)}}dx = 2I(p,q)$	A1
	Since the original integrand was an even function it is also equal to $2I\big(a(p,q), g(p,q)\big)$	E1
		[6]
	$I(x_0,y_0) = I(x_1,y_1) = \cdots = \int_0^\infty \frac{1}{x^2+M^2}dx$	M1
	$= \left[\frac{1}{M}\arctan\left(\frac{x}{M}\right)\right]_0^\infty$	A1
	$= \frac{\pi}{2M}$	A1
		[3]

Question	Answer	Mark
9 i a	Horizontal displacement = d when $t = \frac{d}{v\cos\alpha}$	B1
	Vertical displacement at this time is $$s = v\sin\alpha\left(\frac{d}{v\cos\alpha}\right) - \frac{g}{2}\left(\frac{d}{v\cos\alpha}\right)^2$$	B1
	$d\tan\alpha - \frac{gd^2}{2v^2}\sec^2\alpha > 0$	M1
	$\tan\alpha - \frac{1}{2\lambda}(1+\tan^2\alpha) > 0$	
	$\tan^2\alpha - 2\lambda\tan\alpha + 1 < 0$	M1
	$(\tan\alpha - \lambda)^2 - \lambda^2 + 1 < 0$	
	$(\tan\alpha - \lambda)^2 < \lambda^2 - 1$	A1
	$\lambda^2 - 1 > (\tan\alpha - \lambda)^2 \geq 0$, so $\lambda^2 > 1$	E1
		[6]
b	Horizontal displacement = $2d$ when $t = \frac{2d}{v\cos\alpha}$	M1
	Vertical displacement at this time is $s = v\sin\alpha\left(\frac{2d}{v\cos\alpha}\right) - \frac{g}{2}\left(\frac{2d}{v\cos\alpha}\right)^2 < -2d$	
	$2\tan\alpha - \frac{2}{\lambda}(1+\tan^2\alpha) < -2$	A1
	$\tan^2\alpha - \lambda\tan\alpha - \lambda + 1 > 0$	
	$\left(\tan\alpha - \frac{\lambda}{2}\right)^2 - \frac{\lambda^2}{4} - \lambda + 1 > 0$	M1
	$(2\tan\alpha - \lambda)^2 > \lambda^2 + 4(\lambda - 1)$	A1
	Since $\lambda > 1$, $(2\tan\alpha - \lambda)^2 > \lambda^2$	M1
	$4\tan^2\alpha - 4\lambda\tan\alpha > 0$	
	$\tan\alpha(\tan\alpha - \lambda) > 0$	M1
	Since $\tan\alpha > 0$, $(\tan\alpha - \lambda) > 0$ and so $\tan\alpha > \lambda > 1$ Therefore $\alpha > 45°$	A1
		[7]
ii	To satisfy $\tan^2\alpha - 2\lambda\tan\alpha + 1 < 0$, requires $\frac{2\lambda - \sqrt{4\lambda^2 - 4}}{2} < \tan\alpha < \frac{2\lambda + \sqrt{4\lambda^2 - 4}}{2}$	
	To satisfy $\tan^2\alpha - \lambda\tan\alpha - \lambda + 1 > 0$, requires $\tan\alpha > \frac{\lambda + \sqrt{\lambda^2 + 4(\lambda - 1)}}{2}$	B1
	which is possible provided that $\frac{2\lambda + \sqrt{4\lambda^2 - 4}}{2} > \frac{\lambda + \sqrt{\lambda^2 + 4(\lambda - 1)}}{2}$	M1
	$\lambda + 2\sqrt{\lambda^2 - 1} > \sqrt{\lambda^2 + 4\lambda - 4}$	A1
	Since both sides of the inequality are positive this is equivalent to:	E1
	$\lambda^2 + 4\lambda - 4 < \lambda^2 + 4(\lambda^2 - 1) + 4\lambda\sqrt{\lambda^2 - 1}$	M1
	$1 < \lambda + \sqrt{\lambda^2 - 1}$	A1
	Since $\lambda > 1$ and $\sqrt{\lambda^2 - 1} > 0$ this must be true	A1
		[7]

Question	Answer	Mark
10	Diagram showing the wedge and particle on the plane: There are 4 relevant forces. Weight acting on the particle. Horizontal force P acting on the wedge. Normal force and friction acting on the particle.	 B1 B1
		[2]
i	Resolving forces on the particle (if in equilibrium): $N = mg\cos\alpha$ $F = mg\sin\alpha$ OR $N\sin(\alpha) = F\cos(\alpha)$ (resolving horizontally)	 M1 A1
	$F \le \mu N$, so $\tan\alpha \le \mu$ (so if $\mu < \tan\alpha$ the system cannot be in equilibrium)	 E1
		[3]
ii	For the whole system: $$P = (M+m)a$$ OR for the prism $$P + F\cos\alpha - N\sin\alpha = Ma$$	 B1
	For the particle horizontally and vertically: $N\sin\alpha - F\cos\alpha = ma$ $N\cos\alpha + F\sin\alpha = mg$ OR $N - mg\cos\alpha \ = \ ma\sin\alpha$ $mg\sin\alpha - F \ = \ ma\cos\alpha$	 B1 B1
	So $N\sin\alpha\cos\alpha + F\sin^2\alpha = mg\sin\alpha$ $N\sin\alpha\cos\alpha - F\cos^2\alpha = ma\cos\alpha$	
	$F = \ mg\sin\alpha - \ ma\cos\alpha$	M1
	$$F = m\left(g\sin\alpha - \frac{P\cos\alpha}{M+m}\right) = \frac{m}{m+M}\big((M+m)g\sin\alpha - P\cos\alpha\big)$$	A1
	Also: $N\sin^2\alpha - F\sin\alpha\cos\alpha = ma\sin\alpha$ $N\cos^2\alpha + F\sin\alpha\cos\alpha = mg\cos\alpha$	 M1
	$$N = m\left(\frac{P\sin\alpha}{M+m} + g\cos\alpha\right) = \frac{m}{m+M}\big((M+m)g\cos\alpha + P\sin\alpha\big)$$	A1
		[7]
iii	For equilibrium, require: $-\mu N \le F \le \mu N$	 M1
	$-\tan\lambda\big((M+m)g\cos\alpha + P\sin\alpha\big) \le (M+m)g\sin\alpha - P\cos\alpha$ $\le \tan\lambda\big((M+m)g\cos\alpha + P\sin\alpha\big)$	
	So $$-\tan\lambda\big((M+m)g + P\tan\alpha\big) \le (M+m)g\tan\alpha - P$$ $(M+m)g\tan\alpha - P \le \tan\lambda\big((M+m)g + P\tan\alpha\big)$	 M1
	Therefore: $P(1-\tan\alpha\tan\lambda) \le (M+m)g(\tan\alpha + \tan\lambda)$ $(M+m)g(\tan\alpha - \tan\lambda) \le P(1+\tan\alpha\tan\lambda)$	M1 A1
	Since $\lambda < \frac{1}{4}\pi$, and $\alpha < \frac{\pi}{4}$, $\tan\alpha\tan\lambda < 1$ and so $1 - \tan\alpha\tan\lambda > 0$	 E1
	$P \le \dfrac{(M+m)g(\tan\alpha + \tan\lambda)}{(1-\tan\alpha\tan\lambda)}$ $\dfrac{(M+m)g(\tan\alpha - \tan\lambda)}{(1+\tan\alpha\tan\lambda)} \le P$	 M1 A1
	$(M+m)g\tan(\alpha-\lambda) \le P \le (M+m)g\tan(\alpha+\lambda)$	E1
		[8]

Question	Answer	Mark
11 i	$x^{\frac{1}{x}} = e^{\frac{1}{x}\ln x}$	M1
	$\frac{d}{dx}\left(x^{\frac{1}{x}}\right) = \frac{1-\ln x}{x^2}x^{\frac{1}{x}}$	A1
	$= 0$ when $x = e$	A1
	$\frac{1}{x}\ln x \to 0$ as $x \to \infty$, so $x^{\frac{1}{x}} \to 1$ as $x \to \infty$	B1
	Let $x = \frac{1}{N}$, then $x^{\frac{1}{x}} = \frac{1}{N^N} \to 0$ as $N \to \infty$ / $x \to 0$	B1
	Graph showing the above and no additional turning points. Both coordinates of turning point $\left(e, e^{\frac{1}{e}}\right)$ must be shown.	G1
		[6]
	Since the graph is decreasing for $x > e$, $3^{\frac{1}{3}} > n^{\frac{1}{n}}$ for $n > 3$	E1
	$2^{\frac{1}{2}} = 4^{\frac{1}{4}}$ and so is less than $3^{\frac{1}{3}}$, so the maximum value is $3^{\frac{1}{3}}$.	E1
		[2]
ii	For each group: $P(\text{one test}) = (1-p)^k$	M1
	Expected number of tests is: $1 \cdot (1-p)^k + (k+1)\left(1-(1-p)^k\right)$	M1
	Expected number of tests in total: $r\left(k+1-k(1-p)^k\right)$ $= N\left(1+\frac{1}{k}-(1-p)^k\right)$	A1
		[3]
iii	$N\left(1+\frac{1}{k}-(1-p)^k\right) \le N \Rightarrow \frac{1}{k} \le (1-p)^k$	M1
	$\frac{1}{1-p} \le k^{\frac{1}{k}}$	A1
	By part (i) the maximum value arises when $k = 3$	M1
	and $\frac{1}{1-p} = 3^{\frac{1}{3}}$ $p = 1 - 3^{-\frac{1}{3}}$	A1
	$p - \frac{1}{4} = \frac{3}{4} - 3^{-\frac{1}{3}}$	
	$\left(\frac{3}{4}\right)^{-3} = \frac{64}{27} < 3$, so $3^{-\frac{1}{3}} < \frac{3}{4}$	M1
	So $p - \frac{1}{4} > 0$ and $p > \frac{1}{4}$	A1
		[6]
iv	The term in p^n in the expansion of $(1-p)^k$ is $\frac{k(k-1)\ldots(k-n+1)}{n!}(-p)^n$ $\frac{k(k-1)\ldots(k-n+1)}{n!}p^n < \frac{(kp)^n}{n!}$	M1
	If kp is small: $N\left(1+\frac{1}{k}-(1-p)^k\right) \approx N\left(1+\frac{1}{k}-1+kp\right)$ $= N\left(\frac{1}{k}+kp\right)$	A1
	If $p = 0.01$ and $k = 10$: $\frac{1}{10} + 10(0.01) = \frac{1}{5}$, so the expected number of tests is approximately 20% of N.	E1
		[3]

Question	Answer	Mark
12 i	$P(\text{Value greater than } a) = \dfrac{(n-a)}{n}$	M1
	X is the event that Ada is given number a and all other players are given a number greater than a. $P(X) = \frac{1}{n} \cdot \left(\frac{n-a}{n}\right)^{k-1}$	A1
	If $k = 4$: $P(\text{Ada wins}) = \sum_{a=1}^{n-1} \frac{1}{n} \cdot \left(\frac{n-a}{n}\right)^3 = \frac{1}{n^4} \sum_{a=1}^{n-1} (n-a)^3$	M1
	$= \frac{1}{n^4} \sum_{a=1}^{n-1} a^3$	M1
	$= \frac{(n-1)^2}{4n^2}$ But each player has the same probability of winning, so:	E1
	$P(\text{There is a winner}) = \frac{(n-1)^2}{n^2}$	A1
		[6]
ii	Let $p(a,d)$ be the probability that Ada is given number a, Bob is given number $a+d+1$ and all others are given numbers in between.	
	$p(a,d) = \frac{1}{n^2}\left(\frac{d}{n}\right)^{k-2}$	B1
	$P(\text{Ada has lowest score and Bob has highest score}) = \sum_{d=1}^{n-2} \sum_{a=1}^{n-d-1} \frac{1}{n^2}\left(\frac{d}{n}\right)^2$	M1 A1
	$= \frac{1}{n^4} \sum_{d=1}^{n-2} d^2 \sum_{a=1}^{n-d-1} 1$	M1
	$= \frac{1}{n^4} \sum_{d=1}^{n-2} d^2 (n-d-1) = \frac{1}{n^4} \sum_{d=1}^{n-2} [(n-1)d^2 - d^3]$	M1
	$= \frac{1}{n^4}\left[\frac{(n-1)(n-2)(n-1)(2n-3)}{6} - \frac{(n-2)^2(n-1)^2}{4}\right]$	M1
	$= \frac{(n-1)^2(n-2)}{12n^4}[2(2n-3) - 3(n-2)]$	
	$= \frac{(n-1)^2(n-2)}{12n^3}$	A1
	There are $2 \cdot \binom{4}{2} = 12$ pairs of players and they each have the same probability of winning.	E1
	$P(\text{Two winners}) = \frac{(n-1)^2(n-2)}{n^3}$	A1
		[9]

Question	Answer	Mark
12 ii contd	$P(\text{Winner with lowest number})$ in this game is equal to $P(\text{Winner with highest number})$ and is equal to the probability of there being a winner in the game in part (i)	E1
	$P(\text{Exactly one winner}) = 2 \cdot \frac{(n-1)^2}{n^2} - 2 \cdot \frac{(n-1)^2(n-2)}{n^3}$	M1
	$= \frac{4(n-1)^2}{n^3}$	A1
	Exactly 2 winners more likely if $\frac{(n-1)^2(n-2)}{n^3} > \frac{4(n-1)^2}{n^3}$	M1
	$n - 2 > 4$	
	So $n = 7$ is the minimum	A1
		[5]

This document was initially designed for print and as such does not reach accessibility standard WCAG 2.1 in a number of ways including missing text alternatives and missing document structure.

If you need this document in a different format please email STEPMaths@ocr.org.uk telling us your name, email address and requirements and we will respond within 15 working days.

Cambridge University Press & Assessment
The Triangle Building
Shaftesbury Road
Cambridge
CB2 8EA
United Kingdom